# UNIT 3 VECTORS IN THREE DIMENSIONS

## 3.1 Three-Dimensional Vectors

### 3-DIMENSIONAL SPACE

To this point, we have been working with vectors in 2-dimensional space. Now, we will expand our discussion to 3-dimensional space.

|  |  |
| --- | --- |
| The **2-dimensional coordinate system** is built around a set of two axes that intersect at right angles and one particular point called the origin. Points in the plane are described by ordered pairs and vectors in standard position by . | Diagram showing the 2-dimensional coordinate system with a vector and its coordinates. The vector has a starting point of (0,0) and an ending point of (2,3). There are two black lines that connect the terminal point to the x-axis and the y-axis. |
| The **3-dimensional coordinate system** is built around a set of three axes that intersect at right angles and one particular point again called the origin. Points in the plane are described by ordered triples and vectors in standard position by . | Diagram showing the 3-dimensional coordinate system with a vector and its coordinates. The vector  has a starting point of (0,0,0) and an ending point of (2,3,5). There are two black lines that connect the terminal point to the x-axis, the y-axis, and the z-axis. |

### THE DISTANCE BETWEEN TWO POINTS IN 2 & 3-DIMENSIONAL SPACE

In **two-dimensional space**, the distance between two points say and is given by the distance formula

|  |  |
| --- | --- |
| Diagram showing two points in a 2-dimensional coordinate system and the distance between them. There are two points P and Q where P is located at (x subscript 1, y subscript 1) and Q is located at (x subscript 2, y subscript 2). |  |

In **three-dimensional space**, the distance between two points say and is given by the distance formula

|  |  |
| --- | --- |
|  | Diagram showing two points in a 3-dimensional coordinate system and the distance between them. There are two points P and Q where P is located at ((x subscript 1),(y subscript 1),(z subscript 1)) and Q is located at ((x subscript 2),(y subscript 2),(z subscript 2)). |

The distance between the two points and is

Example (1)

|  |  |
| --- | --- |
| 5.8 units | Diagram showing two points in a 3-dimensional coordinate system and the calculation of the distance between them. There are two points P and Q where P is located at (2,2,5) and Q is located at (5,6,2). There is a line connection P and Q that is (square root of 34) units long. The gray lines connect point P to the x-axis, the y-axis, and the z-axis. The green lines connect point Q to the x-axis, the y-axis, and the z-axis. |

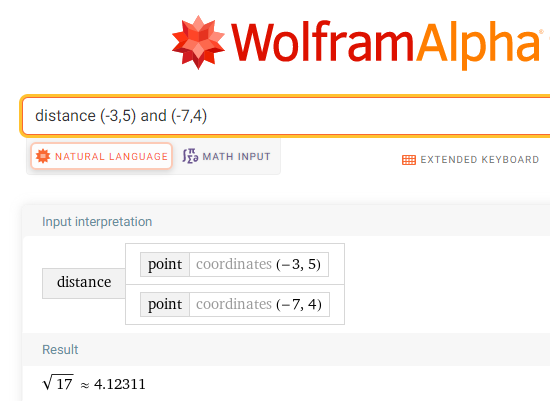
The distance between the two points and is units.

### USING TECHNOLOGY

We can use technology to find the distance between points.

Go to www.wolframalpha.com.

To find the distance between the two points and enter distance and in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, 2311.



### THE EQUATION OF A CIRCLE AND A SPHERE

We can use the distance formulas to get equations of circles and spheres.

The center-radius form of a circle with center at the point and radius is

|  |  |
| --- | --- |
|  | Diagram showing use of distance formula to get equation of a circle with a point and radius. There is a circle with the center point (h,k) with an arrow from the center to the side of the circle labeled r. This circle is inside the xy plane. |

The center-radius form of a sphere with center at the point and radius is

|  |  |
| --- | --- |
|  | Diagram showing use of distance formula to get equation of a sphere with a point and radius. There is a sphere in the xyz plane with the center at (h,k,j). There is an arrow r that connects the center to the edge of the sphere. |

To write the equation of a circle that has the point as its center and radius 8, we use the center-radius form with , , and .

Example (2)

To write the equation of a sphere that has the point as its center and radius 8, we use the center-radius form with , , , and .

Example (3)

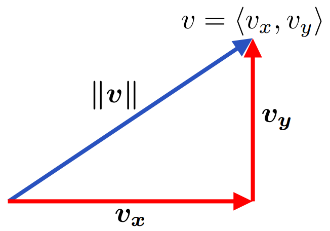
### 3.1 TRY THESE

1. Find the distance between the two points and . Round to one decimal place.
2. Find distance between the two points and . Round to one decimal place.
3. Write the equation of a circle that has the point as its center and radius 1.
4. Write the equation of a sphere that has the point as its center and radius 4.

## 3.2 Magnitude and Direction Cosines of a Vector

### THE MAGNITUDE OF A VECTOR

You likely recall that the magnitude (the length) of a vector in **2-dimensions** is

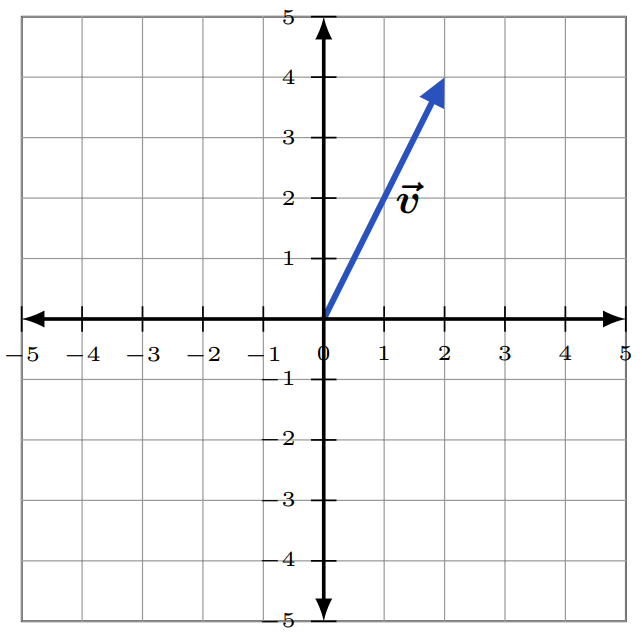


The vector has magnitude

Example (1)

=

Interpret this as the length of the vector is units.

****

The formula for the length of the vector in **3-dimensions** is

The vector has magnitude

Example (2)

=

=

Interpret this as the length of the vector is units.

### THE DIRECTION COSINES OF VECTORS IN 2- AND 3-DIMENSIONS

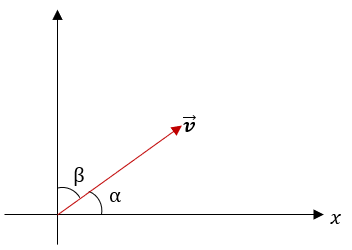
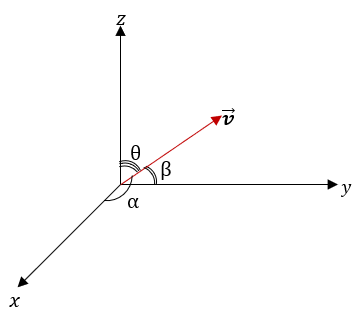
The direction cosines of a vector or are the cosines of the angles the vector forms with the coordinate axes.

The direction cosines are important as they uniquely determine the direction of the vector.

Direction cosines are found by dividing each component of the vector by the magnitude (length) of the vector.

,

, ,

Find the direction cosines of the vector .

Example (3)

First, find the magnitude of the vector

Get the direction cosines by dividing each component, 4, 5, and 2, by this magnitude.

Find the vector that has magnitude 32 and direction cosines

Example (4)

and.

Since ,

, and

.

So, .

### USING TECHNOLOGY

We can use technology to determine the magnitude of a vector.

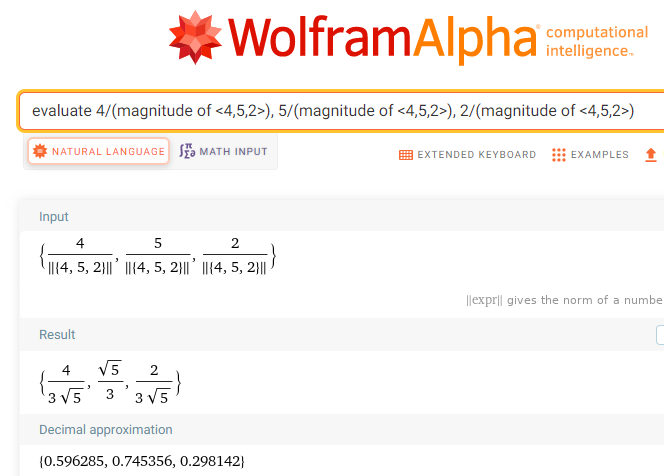
Go to www.wolframalpha.com.

To find the magnitude of the vector enter magnitude of <2, 4, -6> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, .



To find the direction cosines of the vector enter evaluate 4/(magnitude of <4,5,2>), 5/(magnitude of <4,5,2>), 2/(magnitude of <4,5,2>) in the entry field. WolframAlpha answers .

We can use WolframAlpha to approximate a vector give its magnitude and direction cosines.



### 3.2 TRY THESE

1. Find the magnitude of the vector
2. Find the magnitude of the vector
3. Find the cosines of the vector . Round to three decimal places.
4. Approximate the vector that has magnitude 24 and direction cosines  
   .

## 3.3 Arithmetic on Vectors in 3-Dimensional Space

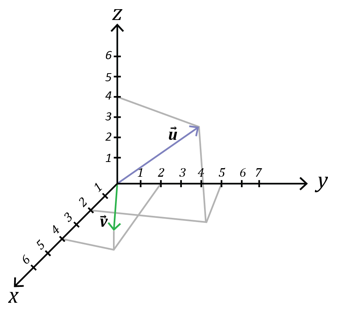
### ADDITION & SUBTRACTION OF VECTORS

To add or subtract two vectors, add or subtract their corresponding components.

To **add** the vectors and add their corresponding components.

Example (1)

So,

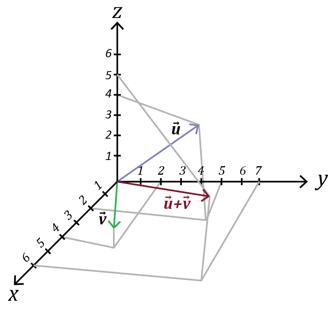


Now, graph this sum. Start at the origin.

Since the component is 6, move 6 units in the direction.

Since the component is 7, move 7 units in the direction.

Since the component is 5, move 5 units upward.



To **subtract** the vectors and subtract their corresponding components.

Example (2)

So,

### SCALAR MULTIPLICATION

Scalar multiplication is the multiplication of a vector by a real number (a scalar).

Suppose we let the letter represent a real number and be the vector Then, the scalar multiple of the vector is

Suppose and .

Example (3)

Then 3

Suppose and .

Example (4)

Then

Suppose , , and . Find

Example (5)

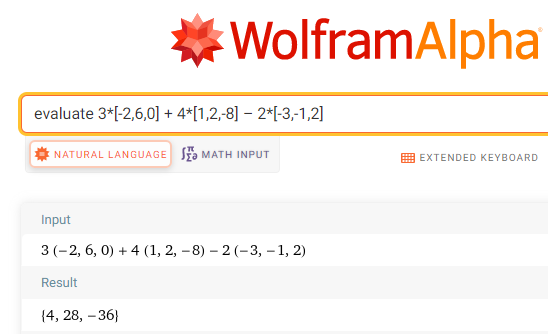
Then

### USING TECHNOLOGY

We can use technology to determine the value of adding or subtracting vectors.

Go to www.wolframalpha.com.

Suppose , , and . Use WolframAlpha to find In the entry field enter evaluate 3\*[-2,6,0] + 4\*[1,2,-8] – 2\*[-3,-1,2].



WolframAlpha answers which is WolframAlpha’s notation for .

### 3.3 TRY THESE

1. Add the vectors
2. Subtract the vector from the vector
3. Given the three vectors, , , and ,  
   find .
4. Suppose , , and , find .

## 3.4 The Unit Vector in 3-Dimensions and Vectors in Standard Position

### THE UNIT VECTOR IN 3-DIMENSIONS

The unit vector, as you will likely remember, in 2-dimensions is a vector of length 1. A unit vector in the same direction as the vector is often denoted with a “hat” on it as in . We call this vector “v hat.”

The unit vector corresponding to the vector is defined to be

The unit vector corresponding to the vector is

Example (1)

A unit vector in 3-dimensions and in the same direction as the vector is defined in the same way as the unit vector in 2-dimensions.

The unit vector corresponding to the vector is defined to be , where

For example, the unit vector corresponding to the vector is

### VECTORS IN STANDARD POSITION

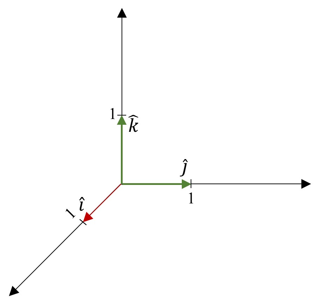
A vector with its initial point at the origin in a Cartesian coordinate system is said to be in *standard position*. A common notation for a unit vector in standard position uses the lowercase letters i, j, and k is to represent the unit vector in

the -direction with the vector , where , and

the -direction with the vector, where , and

the -direction with the vector , where .

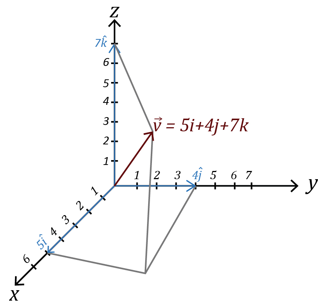
The figure shows these three unit vectors.



Any vector can be expressed as a combination of these three unit vectors.

The vector can be expressed as .

Example (2)



Now, the unit-vector in the direction of is

### NORMALIZING A VECTOR

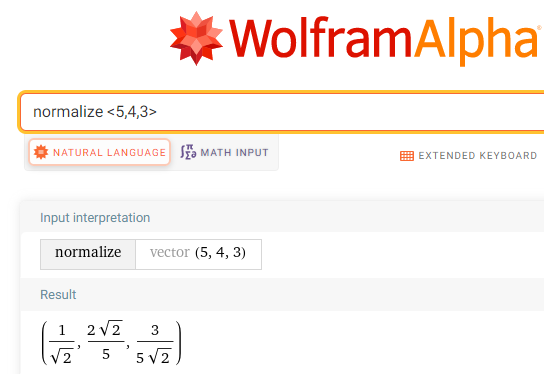
Normalizing a vector is a common practice in mathematics and it also has practical applications in computer graphics. Normalizing a vector is the process of identifying the unit vector of a vector .

### USING TECHNOLOGY

We can use technology to find the unit vector in the direction of the given vector.

Go to www.wolframalpha.com.

Use WolframAlpha to find the unit vector in the direction of . Enter normalize <5,4,3> in the entry field and WolframAlpha gives you an answer.



Translate WolframAlpha’s answer to .

### 3.4 TRY THESE

1. Write the unit vector that corresponds to
2. Write the unit vector that corresponds to .
3. Write the unit vector that corresponds to
4. Normalize the vector .

## 3.5 The Dot Product, Length of a Vector, and the Angle between Two Vectors in Three Dimensions

### THE DOT PRODUCT OF TWO VECTORS

The dot product of two vectors and in two dimensions is nicely extended to three dimensions.

The dot product of vectors and

is a scalar (real number) and is defined to be

Since , , and are real numbers, you can see that the dot product is itself a real number and not a vector.

To compute the dot product of the vectors and ,  
 we compute

Example (1)

Since the dot product is a scalar, it follows the properties of real numbers.

**PROPERTIES OF THE DOT PRODUCT**

1. , the dot product is commutative
2. , the dot product distributes over vector addition
3. , the dot product with the zero vector is the scalar 0.

Compute the dot product , where

Example (2)

, , and ,

### THE LENGTH OF A VECTOR IN THREE DIMENSIONS

The length (magnitude) of a vector in two dimensions is nicely extended to three dimensions.

The dot product of a vector with itself gives the length of the vector.

You can see that the length of the vector is the square root of the sum of the squares of each of the vector’s components. The same is true for the length of a vector in three dimensions.

The dot product of a vector with itself gives the length of the vector.

Use the dot product to find the length of the vector .

Example (3)

In this case, , , and

Using , we get

The length of the vector is units.

### THE ANGLE BETWEEN TWO VECTORS

The formula for the angle between two vectors in two dimensions is nicely extended to three dimensions.

If is the smallest nonnegative angle between two non-zero vectors and , then

or

where and and

Find the angle between the vectors and .

Example (4)

Using , we get

We conclude that the angle between these two vectors is close to 85.7° rounded to one decimal place.

### USING TECHNOLOGY

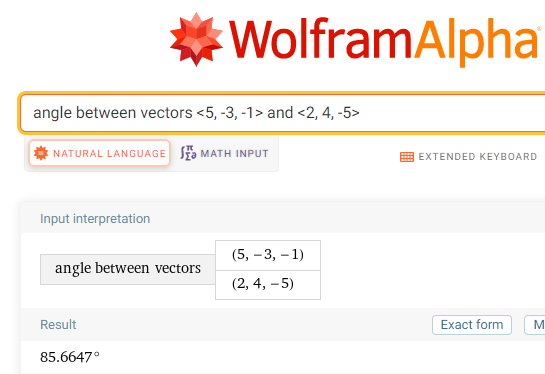
We can use technology to find the magnitude of the vector and the angle between two vectors.

Go to www.wolframalpha.com.

To find the magnitude (length) of the vector enter magnitude of <4, 2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, .



To find the angle between the vectors and ,enter angle between vectors  
<5, -3, -1> and <2, 4, -5> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, , rounded to one decimal place.



### 3.5 TRY THESE

1. Find the dot product of the vectors and .
2. Find the dot product of the vectors and .

1. Find the length of the vector .
2. Find the length of the vector .
3. Find the angle between the vectors and .
4. Find the angle between the vectors and .

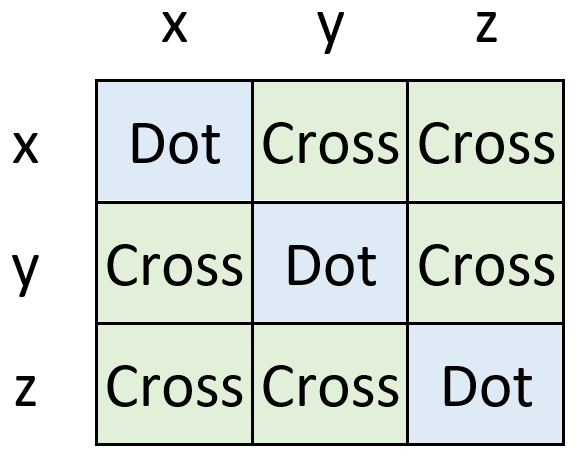
## 3.6 The Cross Product: Algebra

### THE CROSS PRODUCT OF TWO VECTORS

A vector that is perpendicular to both vectors and , can be found using the cross product. The cross product requires that both vectors be in three-dimensional space.

The cross product of vectors and is a vector and is defined to be

This formula is challenging to remember. A nice device to help you remember both this formula and the dot product formula is to visualize them in a 3x3 square of components. The square shows how vectors can interact with one another.



For the cross product,

The -component has a product that involves no -components:

The -component has a product that involves no -components:

The -component has a product that involves no -components:

Each component is a difference of two diagonal products.

|  |  |
| --- | --- |
| Table showing a 3x3 square of components with examples of cross products. The first column is labeled x, the second column is labeled y, and the third column is labeled z. The first row is labeled x, the second row is labeled y and the third row is labeled z. The box for row x, column x says Dot. The box for row x, column y says x*y, and the box for row x, column z says x*z. The box for row y, column x says y*x, the box for row y column y says Dot, and the box for row y column z says y*z. The box for row z column x says z*x, the box for row z, column y says z*y, and the box for row z, column z says Dot. | To produce the -component,  (top right) - (bottom left) = y\*z – z\*y  To produce the -component,  (bottom left) - (top right) = z\*x – x\*z  To produce the -component,  (top right) – (bottom left) = x\*y – y\*x |

The **DOT** product is the interaction between two vectors having **similar** components:

The dot product measures similarity since it combines only interactions of matching components.

The **CROSS** product is the interaction between two vectors having **different** components:

The cross product measures cross interactions since it combines interactions of different components.

Find the cross product of the vectors and .

Example (1)

|  |  |
| --- | --- |
| 3x3 square table with columns labeled 3, 4, negative 7 and rows labeled 5, 2, 4. The box for row 5, column 3 says Dot, the box for row 5, column 4 says 5 multiplied by 4, and the box for row 5 column negative 7 says 5 multiplied by negative 7. The box for row 2, column 3 says 2 multiplied by 3, the box for row 2 column 4 says Dot and the box for row 2 column negative 7 says 2 multiplied by negative 7. The box for row 4 column 3 says 4 multiplied by 3, the box for row 4 column 4 says 4 multiplied by 4, and the box for row 4 column negative 7 says Dot. | 3x3 square table with columns labeled 3, 4, -7 and rows labeled 5, 2, 4. The box for row 5, column 3 says Dot, the box for row 5, column 4 says 20, and the box for row 5 column  -7 says -35. The box for row 2, column 3 says 6, the box for row 2 column 4 says Dot and the box for row 2 column -7 says -14. The box for row 4 column 3 says 12, the box for row 4 column 4 says 16, and the box for row 4 column -7 says Dot. |

To produce the -component, (top right) - (bottom left) = 2\*(-7) – 4\*4 = -30

To produce the -component, (bottom left) - (top right) = 4\*3 – 5\*(-7) = 47

To produce the -component, (top right) – (bottom left) = 5\*4 – 2\*3 = 14

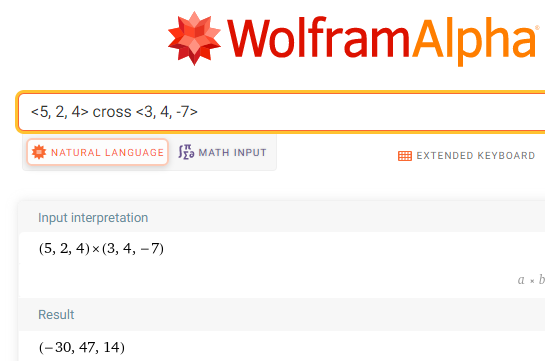
\*Be careful with the computation. It goes (bottom left) – (top right) while  
the first and last go (top right) – (bottom left).

### USING TECHNOLOGY

We can use technology to find the cross product between two vectors.

Go to www.wolframalpha.com.

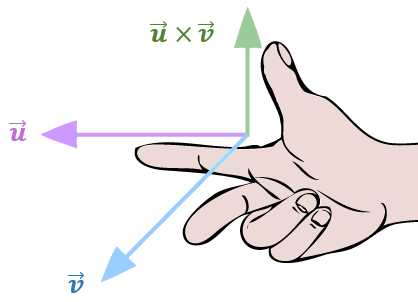
To find the cross product of the vectors and use either the “cross” or the x command. Wolframalpha tells you what it thinks you entered, then it tells you its answer. In this case, .



### THE RIGHT-HAND RULE

You can see that the cross product of the two vectors and , is itself a vector. But where is this vector The cross product of two vectors is a vector that is perpendicular to the plane formed by the two vectors. What about the two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? We use the **right-hand rule**.

Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.



Since the dot product is a scalar, it follows the properties of real numbers.

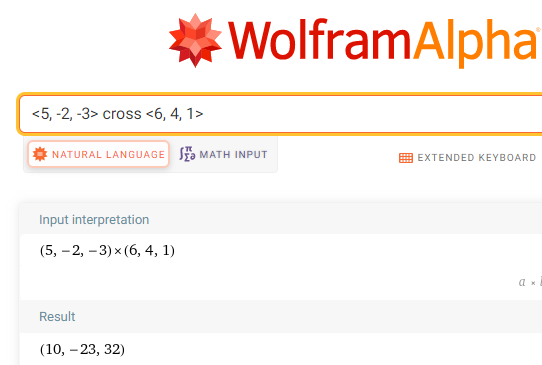
**PROPERTIES OF THE CROSS PRODUCT**

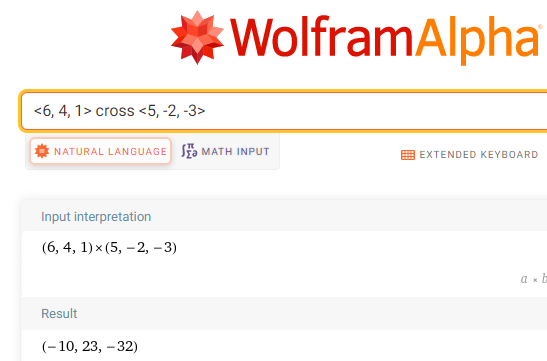
1. , the cross product is **anti-commutative**
2. , the cross product distributes over vector addition
3. =
4. , the cross product with the zero vector is the zero vector

### USING TECHNOLOGY

For example, use WolframAlpha to compute both the cross product and , with

and ,to show that one is the opposite of the other.

****



Notice that , verifying property 1.

### 3.6 TRY THESE

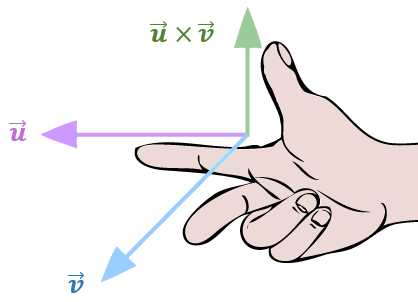
1. Find the cross product of the vectors and .
2. Find the cross product of the vectors and .
3. Find , where , , and .

\* Note that the cross product must be computed first since if it is not, we would be crossing a vector with a scalar.

## 3.7 The Cross Product: Geometry

### THE CROSS PRODUCT OF TWO VECTORS AND THE RIGHT-HAND RULE

The cross product of the two vectors and , is itself a vector. Where is this vector The cross product of two vectors is a vector perpendicular to the plane formed by the two vectors. What if there are two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? Let’s use the **right-hand rule**.

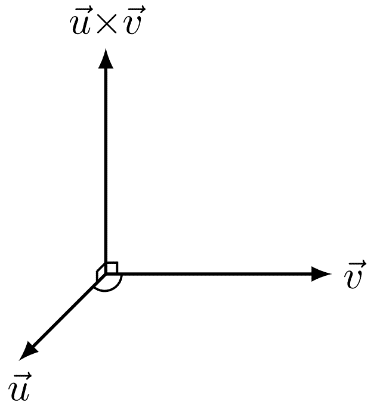


Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.

### THE GEOMETRY OF THE CROSS PRODUCT

If is the angle between the two vectors and , then the length (magnitude) of the cross product is

and



The length of the vector , where and is

Example (1)

We now need to get . We’ll use the formula for the angle between two vectors.

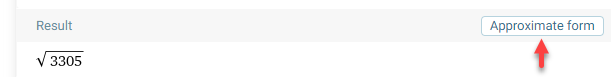
Now we compute units.

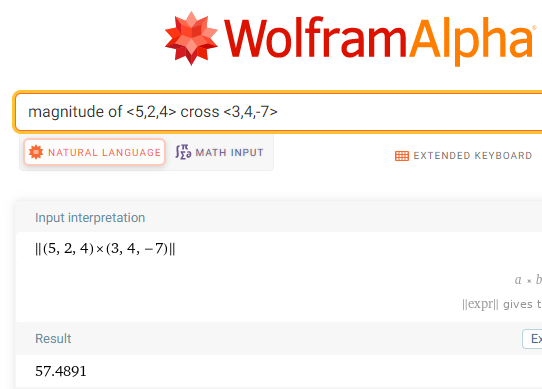
### USING TECHNOLOGY

We can use technology to find the magnitude of the cross product of two vectors.

Go to www.wolframalpha.com.

To find the length of the cross product of the vectors and enter magnitude of <5, 2, 4> cross <3, 4, -7>in the entry field.Wolframalpha tells you what it thinks you entered and its answer. In this case it shows you result of . Click on the approximate form button to get the result in decimal form as .

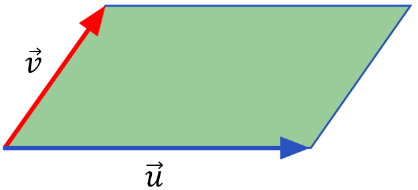




### AREA OF A PARALLELOGRAM

Geometrically, produces the area of a parallelogram determined by and .

Area =



The area of the parallelogram determined by the vectors and from the above example is 57.49 square units.

### THE CROSS PRODUCT OF PERPENDICULAR AND PARALLEL VECTORS

If vectors and are **perpendicular** to each other, then the angle between them is 90° and , so that

If vectors and are **parallel** to each other, then the angle between them is 0° and

It makes sense then to define the cross product of parallel vectors to be the zero vector, . Also, if at least one of the vectors and is the zero vector , then the cross product is defined to be the zero vector. We can say that if the cross product of two vectors is zero, then the vectors are parallel to each other. Also, if two vectors are parallel to each other, then their cross product is zero. We combine these statements together in an *if-and-only-if* statement.

Nonzero vectors and are parallel to each other if and only if .

**PROPERTIES OF THE CROSS PRODUCT**

1. , the cross product is **anti-commutative**
2. , multiplication by a scalar
3. , the cross product distributes over vector addition
4. , the cross product with the zero vector , is the zero vector .

Use WolframAlpha to verify that

Example (2)

where , , and .

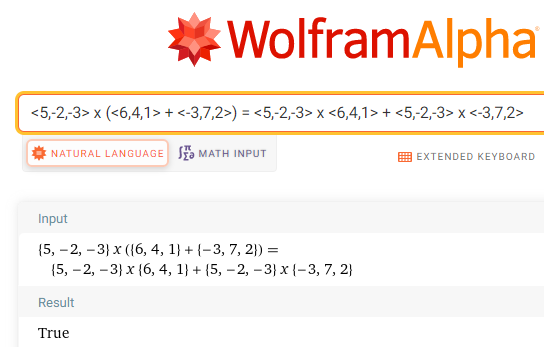
Use W|A to first compute and then . Determine if the results do or do not match. We can do this in one step by entering

<5,-2,-3> x (<6,4,1> + <-3,7,2>) = <5,-2,-3> x <6,4,1> + <5,-2,-3> x <-3,7,2>

If the statement on the left of the = equals the statement on the right, W|A responds with True.

If the statement on the left of the ≠ equals the statement on the right, W|A responds with False.

In this case, we get a True response and have verified the truth of the statement.



### 3.7 TRY THESE

1. Find the cross product of the vectors and .
2. Find the cross product of the vectors and .
3. Find the length of the vector formed by the cross product of the vectors  
    and .
4. Find the angle between the vectors and .
5. Determine if the vectors and are perpendicular or parallel to each other.
6. Find the area of the parallelogram and the triangle formed by the vectors

and .